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A FORMULA FOR THE CASSON INVARIANT BY KAUFFMAN BRACKET SKEIN ALGEBRAS

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ABSTRACT. We give a formula for the Casson invariant using $\zeta' : \mathcal{I}(\Sigma_{g,1}) \to (F^3\mathcal{S}(\Sigma_{g,1})/F^5\mathcal{S}(\Sigma_{g,1}), \mathrm{bch})$, where $(\mathcal{S}(\Sigma), \{F^n\mathcal{S}(\Sigma_{g,1})\}_{n\geq 0})$ is the filtered Kauffman bracket skein algebra of a surface $\Sigma_{g,1}$ of genus g with nonempty connected boundary defined in [2]. Here Let $\mathcal{I}(\Sigma_{g,1})$ be the Torelli group of $\Sigma_{g,1}$.

1. Introduction

Recently it has come to light that the Kauffman bracket skein algebra plays an important role in the study of the relationship between 2-dimensional topology and 3-dimensional topology. We actually define an embedding from the Torelli group of a surface of genus g with nonempty connected boundary into the completed Kauffman bracket skein algebra of the surface in [3]. Furthermore, using this embedding, we construct an invariant for integral homology 3-spheres. In this paper, we give a formula for the Casson invariant, using this construction.

2. Review

We first review some facts about Kauffman bracket skein algebras; for a more detailed treatment, see [1], [2], [3] and [4].

Let Σ be a compact connected oriented surface and I the closed interval $[0,1]$.

2.1. **Definition.** We denote by $\mathcal{T}(\Sigma)$ the set of unoriented framed tangles in $\Sigma \times I$. Let $\mathcal{S}(\Sigma)$ be the Kauffman bracket algebra of Σ, which is the quotient of $\mathbb{Q}[A^{\pm 1}]\mathcal{T}(\Sigma)$ by the skein relation and the trivial knot relation defined by Figure

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1. We denote by $[L]$ the element of $\mathcal{S}(\Sigma)$ represented by $L \in \mathcal{T}(\Sigma)$. The product of $\mathcal{S}(\Sigma)$ is defined by Figure 2. Furthermore, the Lie bracket $[\ ,\ ]$ of $\mathcal{S}(\Sigma)$ is defined by

$$[x, y] = \frac{1}{-A + A^{-1}}(xy - yx).$$

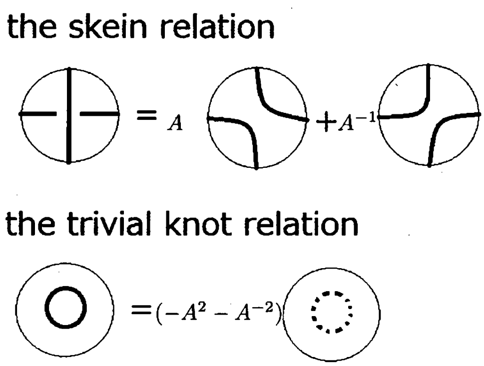

FIG 1. The definition of the skein algebra

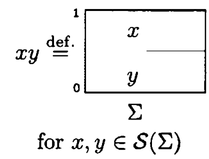

FIG 2. The product

The augmentation map $\epsilon : \mathcal{S}(\Sigma) \to \mathbb{Q}$ is defined by $A \mapsto -1$ and $L \mapsto (-2)^{|L|}$, where $|L|$ is the number of $\pi_0(L)$. For $x \in \pi_1(\Sigma)$, we define $\langle x \rangle \in (\ker \epsilon)/(\ker \epsilon)^2$ by $\langle x \rangle \stackrel{\text{def.}}{=} [L_x] + 2 - 3w(L_x)(A - A^{-1})$ using L_x with the homotopy class of L_x the conjugacy class of $x \in \pi_1(\Sigma) \simeq \pi_1(\Sigma \times I)$, where $w(L_x)$ is the self linking number. The $\mathbb{Q}$-linear map $\lambda : H \wedge H \wedge H \to (\ker \epsilon)/(\ker \epsilon)^2$

$$[a] \wedge [b] \wedge [c] \mapsto \langle abc \rangle - \langle ab \rangle - \langle bc \rangle - \langle ca \rangle + \langle a \rangle + \langle b \rangle + \langle c \rangle$$

is injective where $H \stackrel{\text{def.}}{=} H_1(\Sigma, \mathbb{Q}) = \mathbb{Q} \otimes \pi/[\pi, \pi]$. Let ϖ be the quotient map $\ker \epsilon \to \ker \epsilon/\mathrm{im}\lambda$. We set the filtration $\{F^n\mathcal{S}(\Sigma)\}_{n \geq 0}$ by $F^0\mathcal{S}(\Sigma) \stackrel{\text{def.}}{=} \mathcal{S}(\Sigma)$, $F^1\mathcal{S}(\Sigma) \stackrel{\text{def.}}{=} \ker \epsilon$ and

$$F^{2n}\mathcal{S}(\Sigma) = (\ker \epsilon)^n,$$
$$F^{2n+1}\mathcal{S}(\Sigma) = \ker \varpi (\ker \epsilon)^{n-1}.$$

We remark that

$$[F^n\mathcal{S}(\Sigma), F^m\mathcal{S}(\Sigma)] \subset F^{n+m-2}\mathcal{S}(\Sigma),$$
$$F^n\mathcal{S}(\Sigma)F^m\mathcal{S}(\Sigma) \subset F^{n+m}\mathcal{S}(\Sigma).$$

2.2. **Completion and Torelli group.** We defined the completed skein algebra by

$$\widehat{\mathcal{S}}(\Sigma) \stackrel{\text{def.}}{=} \varprojlim_{n\to\infty}\mathcal{S}(\Sigma)/(\ker\epsilon)^n.$$

We remark that the natural homomorphism $\mathcal{S}(\Sigma) \to \widehat{\mathcal{S}}(\Sigma)$ is injective if $\partial\Sigma \neq \emptyset$. We denote

$$L(c) \stackrel{\text{def.}}{=} \frac{-A+A^{-1}}{4\log(-A)}(\operatorname{arccosh}(-\frac{c}{2}))^2 - (-A+A^{-1})\log(-A).$$

Let $\Sigma_{g,1}$ be a surface of genus g with nonempty connected boundary.

Theorem 2.1 ([3]). *The group homomorphism* $\zeta : \mathcal{I}(\Sigma_{g,1}) \to (F^3\widehat{\mathcal{S}}(\Sigma_{g,1}), \mathrm{bch})$ *defined by* $t_c t_{c'}^{-1} \mapsto L(c) - L(c')$ *for any bounding pair* (c, c') *is well-defined and injective. Here* $\mathrm{bch} : \widehat{\mathcal{S}}(\Sigma) \times \widehat{\mathcal{S}}(\Sigma) \to \widehat{\mathcal{S}}(\Sigma)$ *is the Baker Campbell Hausdorff series as the Lie algebra* $\widehat{\mathcal{S}}(\Sigma)$. *Furthermore, we have* $\zeta(t_c) = L(c)$ *for any null-homologous simple closed curve* c.

2.3. **An invariant for integral homology 3-spheres.** We fix an Heegaard spliting of $S^3 = H_g^+ \cup_\iota H_g^-$ where H_g^+ and H_g^- are handle bodies of genus g and $\iota : \partial H_g^+ \to H_g^-$ is a diffeomorphism. We fix an embedding $\Sigma_{g,1} \hookrightarrow \partial H_g^+$. We denote $H_g^+ \cup_{\iota\circ\xi} H_g^-$ by $M(\xi)$ for an element ξ of the mapping class group of $\Sigma_{g,1}$. Let $e : \Sigma_{g,1} \times I \to S^3$ be the orientation preserving embedding satisfying $e_{|\Sigma_{g,1}\times\{0\}} : \Sigma_{g,1} \times \{0\} \to \Sigma_{g,1}, (t,0) \mapsto t$.

Theorem 2.2. *The map* $Z : \mathcal{I}(\Sigma_{g,1}) \to \mathbb{Q}[[A+1]]$ *defined by*

$$Z(\xi) \stackrel{\text{def.}}{=} \sum_{i=0}^{\infty} \frac{1}{i!(-A+A^{-1})^i} e((\zeta(\xi))^i)$$

induces

$$z : \mathcal{H}(3) \to \mathbb{Q}[[A+1]], M(\xi) \mapsto Z(\xi)$$

where we denote by $\mathcal{H}(3)$ the set of integral homology 3-spheres.

Proposition 2.3. *For $M \in \mathcal{H}(3)$, $z(M) \mod ((A+1)^{n+1})$ is a finite type invariant of order n.*

Proposition 2.4. *For $M \in \mathcal{H}(3)$, the coefficient of $(A+1)$ in $z(M)$ is (-24) times the Casson invariant of M.*

3. The Casson invariant and the Kauffman bracket skein algebra

We fix elements $\alpha_1, \beta_1, \cdots, \alpha_g, \beta_g \in \pi_1(\Sigma_{g,1}, *)$ as shown in Figure 3, where $* \in \partial\Sigma_{g,1}$. We denote the closed curve represented by the conjugacy class of $x \in \pi_1(\Sigma_{g,1}, *)$ by $|x|$. We define a $\mathbb{Q}$-linear map $\rho : S^2(H_1(\Sigma_{g,1}, \mathbb{Q})) \to F^2\mathcal{S}(\Sigma_{g,1})/F^3\mathcal{S}(\Sigma_{g,1})$ by $[x] \cdot [y] \to \langle xy \rangle - \langle x \rangle - \langle y \rangle$ for $x, y \in \pi_1(\Sigma)$, where we denote by $S^2(V)$ the second symmetric tensor of $\mathbb{Q}$-linear space V. We also denote by $\rho : S^2(S^2(H_1(\Sigma_{g,1}, \mathbb{Q}))) \to F^4\mathcal{S}(\Sigma_{g,1})/F^5\mathcal{S}(\Sigma_{g,1})$ define by $\rho(s \cdot t) = \frac{1}{2}(\rho(s)\rho(t) + \rho(s)\rho(t))$ for $s, t \in S^2(H_1(\Sigma_{g,1}, \mathbb{Q}))$. We define the elements of $F^3\mathcal{S}(\Sigma_{g,1})/F^5\mathcal{S}(\Sigma_{g,1})$ as following.

- For $i \neq j$, we set

$$u(i,j) \stackrel{\text{def.}}{=} |\alpha_i\beta_i\alpha_i^{-1}\beta_i^{-1}\alpha_j| - |\beta_j|,$$
$$u'(i,j) \stackrel{\text{def.}}{=} |\alpha_i\beta_i\alpha_i^{-1}\beta_i^{-1}\beta_j| - |\alpha_j|.$$

- For $1 \leq i < j < k \leq g$ and $\epsilon_i, \epsilon_j, \epsilon_k \in \{1, -1\}$, we set

$u(\epsilon_i, \epsilon_j, \epsilon_k, i, j, k) = |\alpha_k^{\epsilon_k}\beta_k\alpha_i^{\epsilon_i}\beta_i| - |\alpha_k^{\epsilon_k}\beta_k\alpha_j^{\epsilon_j}\beta_j\alpha_i^{\epsilon_i}\beta_i(\alpha_j^{\epsilon_j}\beta_j)^{-1}|$.

The group homomorphism $\zeta : \mathcal{I}(\Sigma_{g,1}) \to (F^3\widehat{\mathcal{S}}(\Sigma), \mathrm{bch})$ induces $\zeta' : \mathcal{I}(\Sigma_{g,1}) \to (F^3\mathcal{S}(\Sigma_{g,1})/F^5\mathcal{S}(\Sigma_{g,1}), \mathrm{bch})$. We remark that $\mathrm{bch}(x,y) = x+y+\frac{1}{2}[x,y]$ for $x, y \in F^3\mathcal{S}(\Sigma_{g,1})/F^5\mathcal{S}(\Sigma_{g,1})$.

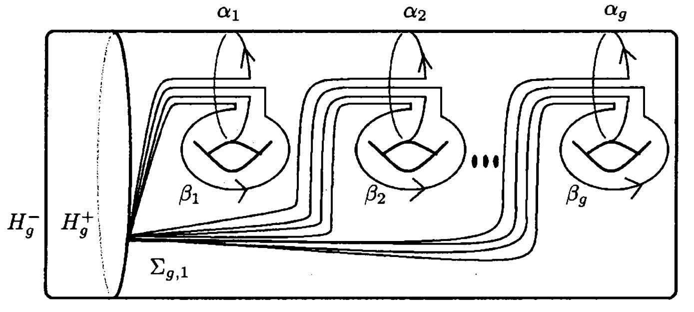

FIG 3. $\alpha_1, \beta_2, \cdots, \alpha_g, \beta_g$

Proposition 3.1. *We have*

$$\begin{aligned}\zeta'(\mathcal{I}(\Sigma_{g,1})) \subset & \mathbb{Q}(\{u(i,j), u'(i,j), u(\epsilon_i, \epsilon_j, \epsilon_k, i, j, k)\}) \\ & + \rho(S^2(S^2(H_1(\Sigma_{g,1}, \mathbb{Q})))) + \mathbb{Q}(A+1)^2.\end{aligned}$$

Theorem 3.2. *Let ξ be an element of $\mathcal{I}(\Sigma_{g,1})$. If*

$$\begin{aligned}\zeta'(\xi) = & \sum_{i<j<k, \epsilon_i, \epsilon_j, \epsilon_k \in \{\pm 1\}} m(\epsilon_i, \epsilon_j, \epsilon_k, i, j, k) u(\epsilon_i, \epsilon_j, \epsilon_k, i, j, k) \\ & + \sum_{i<j} n_{i,j} \rho(([\alpha_i] \cdot [\alpha_j]) \cdot ([\beta_i] \cdot [\beta_j])) \\ & + \sum_i n'_i \rho(([\alpha_i] \cdot [\alpha_i]) \cdot ([\beta_i] \cdot [\beta_i])) \\ & + \sum_i n''_i \rho(([\alpha_i] \cdot [\beta_i]) \cdot ([\alpha_i] \cdot [\beta_i])) \\ & + n'''(A+1)^2 + X,\end{aligned}$$

where X is an element of the subspace of $F^3\mathcal{S}(\Sigma_{g,1})/F^5\mathcal{S}(\Sigma_{g,1})$ generated by

$$\{u(i,j), u'(i,j)\} \cup$$
$$\{\rho((x_1 \cdot x_2) \cdot (x_3 \cdot x_4)) | x_1, x_2, x_3, x_4 \in \{[\alpha_1], \cdots, [\alpha_g], [\beta_1], \cdots, [\beta_g]\}\}$$
$$\backslash \rho(\{([\alpha_i] \cdot [\alpha_j]) \cdot ([\beta_i] \cdot [\beta_j]), ([\alpha_i] \cdot [\alpha_i]) \cdot ([\beta_i] \cdot [\beta_i]), ([\alpha_i] \cdot [\beta_i]) \cdot, ([\alpha_i] \cdot [\beta_i])\}),$$

then the Casson invariant of $M(\xi)$ is

$$\sum_{i<j<k, \epsilon_i, \epsilon_j, \epsilon_k \in \{\pm 1\}} \epsilon_i \epsilon_j \epsilon_j (m(\epsilon_i, \epsilon_j, \epsilon_k, i, j, k))^2$$
$$+ \sum_{i<j} \frac{1}{2} n_{i,j} + \sum_i \frac{3}{4} n'_i + \sum_i n''_i + \frac{1}{48} n'''.$$

Outline of proof. By definition, we have

$$Z(\xi) = 1 + \frac{1}{-A + A^{-1}} e(\zeta(\xi))$$
$$+ \frac{1}{2(-A + A^{-1})^2} e((\zeta(\xi))^2) \mod ((A+1)^2).$$

By straightforward computation, we obtain

$$\frac{1}{-A + A^{-1}} e(\zeta(\xi))$$
$$= (\sum_{i<j} (-12) n_{i,j} + \sum_i (-18) n'_i$$
$$+ \sum_i (-24) n''_i + (-\frac{1}{2} n'''))(A+1) \mod ((A+1)^2),$$

$$\frac{1}{2(-A + A^{-1})^2} e((\zeta(\xi))^2)$$
$$= -24 \sum_{i<j<k, \epsilon_i, \epsilon_j, \epsilon_k \in \{\pm 1\}} \epsilon_i \epsilon_j \epsilon_j (m(\epsilon_i, \epsilon_j, \epsilon_k, i, j, k))^2 (A+1)$$
$$\mod ((A+1)^2).$$

This proves the theorem. $\square$

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